3. INTEGRATION

UNIT 3.1.1 Basics

- **Integration is the reverse of differentiation**

1. Since $\frac{d}{dx}\left(x^2 + c\right) = 2x$, $\int 2x\ dx = x^2 + c$

2. Since $\frac{d}{dx}\left(10x + c\right) = 10$, $\int 10\ dx = 10x + c$

1. Given $\frac{d}{dx}\left(x^3 + c\right) = 3x^2$, find $\int 3x^2\ dx$. $\int 3x^2\ dx = \int\left(\frac{d}{dx}\left(x^3 + c\right)\right)$ $= x^3 + c$	2. Given $\frac{d}{dx}\left(2x^5 + c\right) = 10x^4$, find $\int 10x^4\ dx$.
3. Given $\frac{d}{dx}\left(\frac{1}{2}x^6 + c\right) = 3x^5$, find $\int 3x^5\ dx$.	4. Given $\frac{d}{dx}\left(\frac{2}{x} + c\right) = -\frac{2}{x^2}$, find $\int -\frac{2}{x^2}\ dx$.
5. Given $\frac{d}{dx}\left(4x^3\right) = f(x)$, find $\int f(x)\ dx$.	6. Given $\frac{d}{dx}\left(100x + c\right) = h(x)$, find $\int h(x)\, dx$.
7. Given $\frac{d}{dx}\left(\frac{2}{x-3}\right) = g(x)$, find $\int g(x)\ dx$.	8. Given $\frac{d}{dx}\left(\frac{2x}{x+1}\right) = f(x)$, find $\int f(x)\ dx$.
9. Given $\frac{d}{dx}\left(x(x+2)^3 dx\right) = w(x)$, find $\int w(x)\ dx$.	10. Given $\frac{d}{dx}\left[(x-3)(x+2)\right] = p(x)$, find $\int p(x)\ dx$.

UNIT 3.1.2 (a) Integration of x^n :

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c,\quad n \neq -1$$

1. $\int x^3\ dx = \frac{x^{3+1}}{3+1} + c$ $= \frac{x^4}{4} + c$	2. $\int x^5\ dx =$	3. $\int x^9\ dx =$
4. $\int x^{-3}\ dx =$	5. $\int x^{-2}\ dx =$	6. $\int x\ dx =$

UNIT 3.1.2 (b) Integration of ax^n :

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c \,, \quad n \neq -1$$

Note : $\int m \, dx = mx + c$, m a constant

1. $\int 6x^3 \, dx = 6.\frac{x^{3+1}}{3+1} + c$ $= 6.\frac{x^4}{4} + c$ $= \frac{3x^4}{2} + c$	2. $\int 10x^4 \, dx =$	3. $\int 4x^3 \, dx =$
4.. $\int 10 \, dx =$ ***10x + c***	5. $\int \frac{1}{2} dx =$	6.. $\int -3dx =$
7. $\int 8x \, dx = 8.\frac{x^{1+1}}{1+1} + c$ $= 8.\frac{x^2}{2} + c$ $= 4x^2 + c$	8. $\int 6x \, dx =$	9. $\int 3x \, dx =$
10. $\int 12x^3 \, dx =$	11. $\int 8x^2 \, dx =$	12. $\int 10\,x^5 \, dx =$
13. $\int \frac{2}{x^3} \, dx = \int 2x^{-3} \, dx$ $= 2.\frac{x^{-3+1}}{-3+1} + c$ $= 2.\frac{x^{-2}}{-2} + c$ $= -\frac{1}{x^2} + c$	14. $\int \frac{8}{x^5} \, dx = \int 8x^{-5} \, dx$ $=$	15. $\int \frac{12}{x^4} \, dx =$
16. $\int \frac{2}{5x^3} \, dx =$	17. $\int \frac{2}{3} x \, dx =$	18. $\int 0.9x^2 \, dx =$

UNIT 3.1.3 To Determine Integrals of Algebraic Expressions.

Note : Integrate term by term. Expand & simplify the given expression where necessary.

Example : $\int (3x^2 - 4x + 5)dx = \dfrac{3x^3}{3} - \dfrac{4x^2}{2} + 5x + c$

$= x^3 - 2x^2 + 5x + c$

1. $\int (6x - 4)dx$ =	2. $\int (12x^2 + 8x - 1)dx$ =	3. $\int (x^3 - 3x + 2)dx$ =
4. $\int x(3x - 2)dx$ =	5. $\int (2x - 1)(2x+1)dx$ =	6. $\int (x+2)(x-3)dx$ =
7. $\int (3x - 2)^2 dx$ =	8. $\int \dfrac{(2x - 1)(2x+1)}{x^2} dx$ =	9. $\int \dfrac{6x^2 - 4}{x^2} dx$ =
10. $\int \dfrac{(3x + 4)^2}{x^2} dx$ =	11. $\int (2x^{-2} - x + 1)dx$ =	12. $\int (2 - x)^2 dx$ =
13. $\int (3-\frac{1}{x})(3+\frac{1}{x})dx$ =	14. $\int (-2x^{-2} + 3 - x + \frac{1}{x^2})dx$ =	15. $\int x(3 - x)^2 dx$ =

UNIT 3.1.4 To Determine the Constant of Integration (I)

	Example		Exercise
1	***SPM 2003, K2, S3(a) 3 Marks*** *Given* $\frac{dy}{dx} = 2x + 2$ and $y = 6$ when $x = -1$, find y in terms of x. *Answer:* $\frac{dy}{dx} = 2x + 2$ $y = \int (2x+2)\,dx$ $= \frac{2x^2}{2} + 2x + c$ $y = x^2 + 2x + c$ $y = 6, x = 1,$ $6 = 1^2 + 2(1) + c$ $6 = 3 + c$ $c = 3$ Hence $\boldsymbol{y = x^2 + 2x + 3}$	2	Given $\frac{dy}{dx} = 2x + 3$ and $y = 4$ when $x = 1$, find y in terms of x. $y = x^2 + 3x$
3	Given $\frac{dy}{dx} = 4x + 1$ and $y = 4$ when $x = -1$, find y in terms of x. *Answer :* $\frac{dy}{dx} = 4x + 1$ $y = \int (4x+1)\,dx$ $=$ $y = 2x^2 + x + 3$	4	Given $\frac{dy}{dx} = 6x - 3$ and $y = 3$ when $x = 2$, find y in terms of x. $y = 3x^2 - 3x - 3$
5	Given $\frac{dy}{dx} = 4 - 2x$ and $y = 5$ when $x = 1$, find y in terms of x. $y = 4x - x^2 + 2$	6	Given $\frac{dy}{dx} = 3x^2 - 2$ and $y = 4$ when $x = -1$, find y in terms of x. $y = x^3 - 2x + 3$
7	Given $\frac{dy}{dx} = 3(2x - 4)$ and $y = 30$ when $x = -2$, find y in terms of x. $y = 3x^2 - 12x - 6$	8	Given $\frac{dy}{dx} = 2 - \frac{3}{x^2}$ and $y = 1$ when $x = -1$, find y in terms of x. $y = 2x + \frac{3}{x} + 6$

UNIT 3.1.4 To Determine the Constant of Integration (II)

<table>
<tr><th colspan="2">Example</th><th colspan="2">Exercise</th></tr>
<tr><td>1</td><td>

Given $\frac{dy}{dx} = 6x - 4$ and $y = 2$ when $x = -1$, find the value of y when x = 2.

Answer: $\frac{dy}{dx} = 6x - 4$

$$y = \int (6x-4)\,dx$$

$$= \frac{6x^2}{2} - 4x + c$$

$$y = 3x^2 - 4x + c$$

$y = 2, x = -1$, $2 = 3(-1)^2 - 4(-1) + c$

$2 = 3 + 4 + c$

$c = -5$

$\boldsymbol{y = 3x^2 - 4x - 5}$

When x = 2, $\boldsymbol{y = 3(2)^2 - 4(2) - 5 = -1}$

</td><td>2</td><td>

Given $\frac{dy}{dx} = 2x + 5$ and y = 1 when x = – 1 , find the value of y when x = 3.

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</td></tr>
<tr><td>3</td><td>

Given $\frac{dy}{dx} = 1 - 3x^2$ and y = 2 when x = -1, find the value of y when x = 2.

Answer: $\frac{dy}{dx} = 1 - 3x^2$

$$y = \int (1 - 3x^2)\,dx$$

=

– 4

</td><td>4</td><td>

Given $\frac{dy}{dx} = 6x - 3$ and y = 3 when x = 2, find the value of y when x = 1.

– 3

</td></tr>
<tr><td>5</td><td>

Given $\frac{dy}{dx} = 4 - 2x$ and y = 5 when x = 1 find the value of y when x = – 1.

-3

</td><td>6</td><td>

Given $\frac{dy}{dx} = 3x^2 - 2$ and y = 4 when x = -1, find the value of y when x = 0.

3

</td></tr>
<tr><td>7</td><td>

Given $\frac{dy}{dx} = 3(2x - 1)$ and y = 5 when x = -2, find the value of y when x = 1.

-13

</td><td>8</td><td>

Given $\frac{dy}{dx} = 5 - \frac{3}{x^2}$ and y = 1 when x = – 1 , find the value of y when x = 3

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</td></tr>
</table>

3.1.5 To Determine the Equation of Curve from Gradient Function

<table>
<tr><th colspan="2">Example</th><th colspan="2">Exercise</th></tr>
<tr><td>1</td><td>SPM 2004, K2, S5 (3 Marks)
The gradient function of a curve which passes through A(1, -12) is $3x^2 - 6x$.
Find the equation of the curve.
Answer: $\frac{dy}{dx} = 3x^2 - 6x$
$y = \int (3x^2 - 6x)\,dx$
$= \frac{3x^3}{3} - \frac{6x^2}{2} + c$
$y = x^3 - 3x^2 + c$
$y = -12, x = 1,$ $-12 = 1^3 - 3(1) + c$
$-12 = -2 + c$
$c = -10$
Hence $y = x^3 - 6x - 10$</td><td>2.</td><td>The gradient function of a curve which passes through B(1, 5) is $3x^2 + 2$.
Find the equation of the curve.
Answer: $\frac{dy}{dx} = 3x^2 + 2$
$y = \int (3x^2 + 2)\,dx$
=

$y = x^3 + 2x + 2$</td></tr>
<tr><td>3.</td><td>The gradient function of a curve which passes through P(1, – 3) is $4x - 6$.
Find the equation of the curve.

Answer:

$y = 2x^2 - 6x + 1$</td><td>4.</td><td>The gradient function of a curve which passes through Q(– 1 , 4) is $3x (x - 2)$.
Find the equation of the curve.

Answer:

$y = x^3 - 3x^2 + 8$</td></tr>
<tr><td>5.</td><td>The gradient function of a curve which passes through A(1, 6) is $5 - 3x^2$.
Find the equation of the curve.

$y = 5x - x^3 + 2$</td><td>6.</td><td>The gradient function of a curve which passes through R(2, 3) is $6x^2 - 4$.
Find the equation of the curve.

$y = x^3 - 4x + 3$</td></tr>
<tr><td>7.</td><td>The gradient function of a curve which passes through A(1, 10) is $x(6 - 3x)$.
Find the equation of the curve.

$y = 3x^2 - x^3 + 8$</td><td>8.</td><td>The gradient function of a curve which passes through A(-1, 7) is $3x^2 + 2x - 1$.
Find the equation of the curve.

$y = x^3 + x^2 - x + 6$</td></tr>
</table>

UNIT 3.1.6 Integration of the form $\int (ax+b)^n\,dx,\ n \neq -1$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\ ,\ n \neq -1$$

E X A M P L E	$\int (3x+2)^4\,dx = \frac{(3x+2)^{4+1}}{3(4+1)} + c$ $= \frac{(3x+2)^5}{15} + c$	E X A M P L E	$\int \frac{12}{(2x-3)^4}\,dx = \int 12(2x-3)^{-4}\,dx$ $= \frac{12(2x-3)^{-3}}{-3.\ (2)} + c$ $=$
1.	$\int (2x+5)^3\,dx =$	2.	$\int \frac{4}{(x-3)^2}\,dx =$
3.	$\int (2+4x)^5\,dx =$	4.	$\int (x+2)^{-4}\,dx =$
5.	$\int \frac{3}{(2x-1)^2}\,dx =$	6.	$\int \frac{15}{(3x-5)^6}\,dx =$
7.	$\int 6(2-x)^3\,dx =$	8.	$\int 30(4+3x)^{-3}\,dx =$
9.	$\int \frac{2}{3}(1-2x)^3\,dx =$	10.	$\int \frac{15}{2(x-3)^4}\,dx =$

UNIT 3.2 Definite Integral

$$If \int f(x)dx = F(x)+c, \quad then \quad \int_a^b f(x)dx = F(b)-F(a)$$

1. $\int_1^2 2x\ dx = \left[x^2\right]_1^2$ $= \mathbf{2^2 - 1^2}$ $= \mathbf{4 - 1}$ $= \mathbf{3}$	2. $\int_0^3 4x^3\ dx = \left[x^4\right]_0^3$ $=$ *[81]*	3. $\int_1^2 6x^2\ dx =$ $=$ [14]
4. $\int_1^2 8x\ dx =$ $=$ *[12]*	5. $\int_2^4 x^3\ dx =$ $=$ *[60]*	6. $\int_1^3 10\ dx =$ $=$ *[20]*
7. $\int_1^2 3x^{-2}\ dx =$ $=$ *[* $\frac{3}{2}$ *]*	8. $\int_1^3 (\frac{2}{x^3})\ dx =$ *[* $\frac{8}{9}$ *]*	9. $\int_1^2 \frac{3}{2x^2}\ dx =$ $=$ *[* $\frac{3}{4}$ *]*
10. $\int_0^3 (2-6x)\ dx =$ *[-21]*	11. $\int_1^3 (4x-3x^2)\ dx =$ *[-10]*	12. $\int_0^3 x(2x+1)\,dx =$ *[22.5]*
13. $\int_1^2 (2x-1)(2x+1)\,dx$ $=$ [$\frac{25}{3}$]	14. $\int_1^3 (3x-2)^2\ dx$ $=$ *[38]*	15. $\int_0^1 x(3x-2)\,dx =$ *[0]*

3.2.1 Definite Integral of the form $\int (ax+b)^n\,dx,\ n \neq -1$

E X A M P L E	$\int_0^1 (3x+2)^4\,dx = \left[\frac{(3x+2)^5}{5.3}\right]_0^1$ $= \left[\frac{(3x+2)^5}{15}\right]_0^1$ $= \frac{5^5}{15} - \frac{2^5}{15}$ $= 206.2$	You Try!	$\int_1^2 \frac{3}{(2x-1)^2}\,dx = \int_1^2 3(2x-1)^{-2}\,dx$ $=$ $=$ *[1]*
1.	$\int_0^1 (2x+3)^2\,dx =$ *[$16\frac{1}{3}$]*	2.	$\int_4^5 \frac{10}{(x-3)^2}\,dx =$ *[5]*
3.	$\int_0^1 16(2+4x)^3\,dx =$ *[1280]*	4.	$\int_0^1 6(x+2)^{-3}\,dx =$ *[$\frac{5}{12}$]*
5.	$\int_1^2 \frac{6}{(2x-1)^2}\,dx =$ *[2]*	6.	$\int_2^3 \frac{24}{(3x-5)^3}\,dx =$ *[3.75]*

UNIT 3.2.3 Applications of Definite Integral (I)

Given that $\int_1^3 f(x)dx = 4$, $\int_1^3 g(x)dx = 6$. Find the value of			
1.	$\int_3^1 f(x)dx =$ *-4*	2.	$\int_1^3 [2 + f(x)]dx =$ 8
3.	$\int_1^3 [5f(x)]dx =$ 20	4.	$\int_1^3 [f(x) - 4]dx =$ -4
5.	$\int_1^3 [4f(x) - 2x]dx =$ 8	6.	$\int_1^3 \left[\frac{3f(x)-1}{2}\right]dx =$ 5
7.	$\int_1^3 [3f(x) - 2g(x)]dx =$ 0	8.	$\int_1^3 \left[2g(x) + \frac{1}{2}f(x)\right]dx =$ 14
9.	$\int_1^3 \frac{[6f(x) - g(x)]}{3}dx =$ 6	10.	$\int_1^3 \left[\frac{3}{2}f(x) + 2g(x) - 1\right]dx =$ 16

UNIT 3.2.3 Applications of Definite Integral (II)

1.	Given that $\int_{-1}^{k}(2x-3)dx = 6$, where k > -1, find the value of k. *(SPM 2004, P1, Q 22)* *k = 4*	2.	Given that $\int_{0}^{k}(2x-1)dx = 12$, where k > 0, find the value of k. 4
3.	Given that $\int_{0}^{k}(3-4x)dx = -20$, where k > 0, find the value of k. 4	4.	Given that $\int_{0}^{k}(6x+1)dx$ = 14 , where k > 0, find the value of k. 2
5.	Given that $\int_{2}^{6} f(x) = 7$ and $\int_{2}^{6}\left(2f(x)-kx\right)=10$, find the value of k. [SPM '05, P1,Q21] ¼	6.	Given that $\int_{1}^{3} f(x) = 5$ and $\int_{1}^{3}\left(3f(x)+kx\right)=23$, find the value of k. 2
5.	Given that $\int_{2}^{5} f(x) = 6$ and $\int_{2}^{5}\left(f(x)+kx\right)=10$, find the value of k. $\frac{8}{21}$	6.	Given that $\int_{1}^{4} f(x) = 3$ and $\int_{1}^{4}\left(kf(x)+6x\right)=39$, find the value of k. *[k=-2]* -2

Questions based on SPM Format (I)

1.	Find the value of $\int_1^2 \frac{3-x^2}{x^2}dx$. ½	2	Find the value of $\int_0^3 x(4-x)\,dx$. 9
3.	Find the value of $\int_0^2 \frac{5dx}{(1+2x)^2}$. 2	4	Find the value of $\int_1^4 \frac{15}{(3x-2)^2}dx$. 4.5
5.	Given that $\int_1^k (2-2x)dx = -16$, where k > 0, find the value of k. ***k = 5***	6.	Given that $\int_0^k (4x-1)dx = 6$, where k > 0, find the value of k. *k = 2*
7.	Given that $\int_0^4 f(x) = 15$ and $\int_0^4 \left(2f(x)-kx\right) = 6$, find the value of k. k =3	8.	Given that $\int_1^4 f(x) = 10$ and $\int_1^4 \left(3f(x)-kx\right) = 0$, find the value of k. k =4

SPM FORMAT QUESTIONS (II)

1.	*(SPM 06, P1,Q21)* Given that $\int_1^5 g(x)dx = 8$, find (a) the value of $\int_5^1 g(x)dx$, (b) the value of k if $\int_1^5 [kx - g(x)]dx$ = 10. (a) -8 (b) $\frac{3}{2}$	2.	Given that $\int_1^4 g(x)dx = 10$, find (a) the value of $\int_4^1 2g(x)dx$, (b) the value of k if $\int_1^4 [k + 2g(x)]dx$ = 50. (a) -20 (b) k = 10
3.	Given that $\int_0^3 g(x)dx = 5$, find the value of p if $\int_0^3 [2px + 3g(x)]dx = -3$. p = 2	4.	Given that $\int_1^4 g(x)dx = 9$, find the value of p if $\int_1^4 [kx - 3g(x)]dx$ = 48. *k = 10*
5.	*Given* $\frac{dy}{dx}$ = 4*x* + 2 and *y* = 10 when *x* = – 1, find y in terms of x. *y = 2x² +2x + 10*	6.	Given $\frac{dy}{dx}$ = $1 - 2x + 3x^2$ and y = 3 when x = 1, find y in terms of x. *y = x – x² + x³ + 2*
7.	The gradient function of a curve which passes through P(1, 3) is $8x - 3x^2$. Find the equation of the curve. *y = 8x – 3x²*	8.	The gradient function of a curve which passes through Q(-1, 4) is $2x - \frac{1}{x^2}$. Find the equation of the curve. $y = x^2 + \frac{1}{x} + 4$

SPM FORMAT QUESTIONS (III)

1. *(SPM 06, P1,Q20)*

Diagram 8 shows the curve $y = f(x)$ cutting the x-axis at $x = a$ and $x = b$.

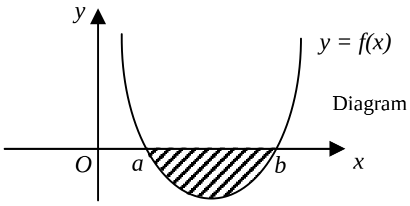

Diagram 8

Given that the area of the shaded region is 5 unit^2, find the value of $\int_a^b 2f(x)dx$.

Answer : ....

2. Diagram 9 shows the curve $y = f(x)$ cutting the x-axis at $x = a$ and $x = b$.

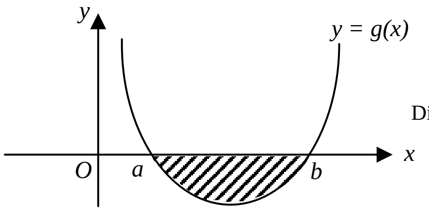

Diagram 9

Given that the area of the shaded region is **6** unit^2, find the value of $\int_b^a 3g(x)dx$.

Answer : ....

3. Diagram 10 shows part of the curve $y = f(x)$.

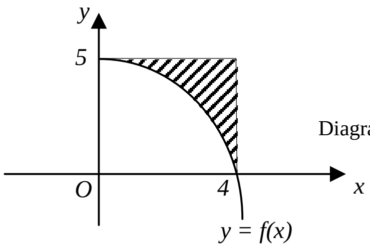

Diagram 10

Given that $\int_0^4 f(x)dx$ = 15 unit^2, find the area of the shaded region.

Answer : ....

4. Diagram 11 shows part of the curve $y = f(x)$.

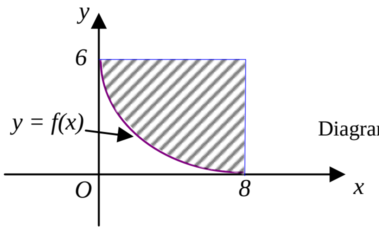

Diagram 11

Given that the area of the shaded region is 40 unit^2, find the value of $\int_0^8 f(x)dx$.

Answer : ....

5. *(SPM 01)* Diagram 12 shows the sketch of part of a curve.

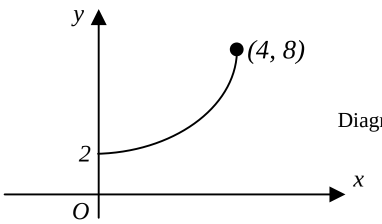

Diagram 12

(a) Shade, on the given diagram, the region represented by $\int_2^8 x\,dy$.

(b) Find the value of $\int_0^4 y\,dx + \int_2^8 x\,dy$

Answer : (b) ....

6. Diagram 13 shows the sketch of part of a curve.

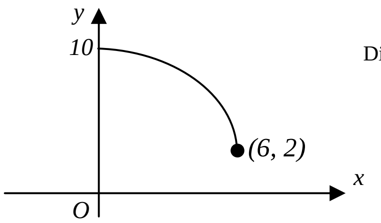

Diagram 13

(a) Shade, on the given diagram, the region represented by $\int_2^{10} x\,dy$.

(b) If $\int_2^{10} x\,dy$ = p , find , in terms of p, the value of $\int_0^6 y\,dx$.

Answer : (b) ....

SPM FORMAT QUESTIONS (IV)

1.	*(SPM'03)* Given that $\int \frac{5}{(1+x)^4}\,dx$ = k(1+ x)^n + c , find the value of k and n. [3 marks] $[k = -\frac{5}{3},\ n = -3]$	2.	Given that $\int \frac{12}{(3x-2)^3}\,dx$ = k(3x – 2)^n + c , find the value of k and n. k = -2 , n = -2
3.	*(SPM 02)* Given that $4x + 3\frac{dy}{dx} = 6$, express y in terms of x. $y = 2x - \frac{2}{3}x^2 + c$	4.	Given that $8x - 2\frac{dy}{dx} = 5$, express y in terms of x. $y = 2x^2 - \frac{5}{2}x + c$
5.	*(SPM 1999)* Given that $\frac{d^2y}{dx^2} = 5x^4 + 1$, when x = 1, y = – 3 and $\frac{dy}{dx}$ = – 2 , find y in terms of x. $y = \frac{1}{6}x^6 + \frac{1}{2}x^2 - 4x + \frac{1}{3}$	6.	Given that $\frac{d^2y}{dx^2} = 2 - 6x$, when x = – 1 , y = 3 and $\frac{dy}{dx}$ = – 1 , find y in terms of x. $y = x^2 - x^3 + 4x + 5$

7.	(SPM 2002) Given that $\int_0^1 \left(16x^2 + 10kx + k^2\right) dx = \frac{4}{3}$. Find the possible values of k. *k = -1 , -4*	8.	Given that $\int_0^1 \left(3x^2 + 10kx + 4\right) dx = 0$. Find the value of k. k = -1
9.	*(SPM 01)* Given that $\frac{d}{dx}\left(\frac{x^2}{x-1}\right) = g(x)$, find the value of $\int_2^3 \left(2x - g(x)\right) dx$. . $\frac{9}{2}$	10	Given that $\frac{d}{dx}\left(\frac{x}{x-1}\right) = f(x)$, find the value of $\int_2^3 \left(4x + f(x)\right) dx$. . $13\frac{1}{2}$
11.	Given that $\frac{d}{dx}\left(\frac{x-1}{x+2}\right) = h(x)$, find the value of $\int_0^1 \left(6 - 4h(x)\right) dx$. [4]	12	Given that $\frac{d}{dx}\left(\frac{12}{(x+1)^3}\right) = w(x)$, find the value of $\int_0^1 \left(6\,w(x) - 3\right) dx$. . [– 66]